Exam

23/01/2025, 8:30 am - 10:30 am

Instructions:

- Prepare your solutions in an **ordered, clear and clean way**. Avoid delivering solutions with scratches.
- Write your name and student number in **all** pages of your solutions.
- Clearly indicate each exercise and the corresponding answer. Provide your solutions with as much detail as possible.
- You are allowed to use one and only one A4 cheat sheet.

Exercise 1: Consider the function on $\mathbb{R}^3$ defined by

$$f(x,y,z) = \begin{cases} \dfrac{xyz}{x^4+y^4+z^4} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0). \end{cases}$$

(a) (1 point) Show that all partial derivatives exist everywhere.

(b) (1 point) Where is f differentiable? (justify your answer sufficiently)

(c) (0.5 points) Is f continuous at the origin? (justify your answer sufficiently)

Exercise 2: (1 point) Does the system of equations

$$\begin{cases} x+y+\sin(xy) & = a \\ \sin\left(x^2+y\right) & = 2a \end{cases}$$

have a solution for sufficiently small a? (justify your answer sufficiently)

Exercise 3: (1 point) Let A be the region in $\mathbb{R}^2$ bounded by $\{x=0\}, \{y=0\}$, and $\{x+y=2\}$. Compute $\int_A xy^2 dxdy$.

Exercise 4: Consider the torus in $\mathbb{R}^3$ defined parametrically as

$$\phi(u,v) = ((2+\cos v)\cos u, (2+\cos v)\sin u, \sin v), \quad u,v \in [0,2\pi].$$

(a) (1.5 points) Is the given parametrization orientation-preserving if the torus is oriented by the outward normal vector?

(b) (1.5 points) Evaluate the surface integral of the 2-form $\omega = x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y$ over the torus.

Exercise 5: A vector field $\vec{F}$ is called conservative if there exists a scalar potential function $\phi(x,y,z)$ such that $\vec{F} = \nabla\phi$.

(a) (0.5 points) Let $\vec{F}(x,y,z) = (yz, xz, xy)$ be a vector field defined on $\mathbb{R}^3$. Is $\vec{F}$ conservative?

(b) (0.5 points) Let $\vec{F}(x,y,z) = (yz, xz, xy)$ be a vector field defined on $\mathbb{R}^3 \setminus \{0\}$. Is $\vec{F}$ conservative?

Exercise 6: (1.5 points) Let A be a region in the first octant $x,y,z \geq 0$, where $z \leq 4$ and $x^2+y^2 \leq 4$. Let S be the boundary of A, oriented by the outward-pointing normal. What is the flux of $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x+yz \\ y+xz \\ z+xy \end{bmatrix}$ across S?

Bonus questions:

Exercise 7: (1 point) Let $\vec{F}$ be the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \\ 0 \end{bmatrix}$, where F_1 and F_2 are defined on all of $\mathbb{R}^2$.

Find conditions on F_1 and F_2 that guarantee that there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \nabla f$.

Exercise 8: (1 point) Let C be the curve in the (x,y) plane given by the polar equation $r^2 = 2\cos(2\theta)$. Orient this curve by increasing θ and integrate over the curve the 1-form $\dfrac{-y\mathrm{d}x + (x+1)\mathrm{d}y}{(x+1)^2+y^2} + \dfrac{-(y+1)\mathrm{d}x + x\mathrm{d}y}{x^2+(y+1)^2}$.